2001 ADS Seminar

"Trajectory Reconstruction & Measurement Techniques"

Topics:

Introduction

Trajectory Reconstruction

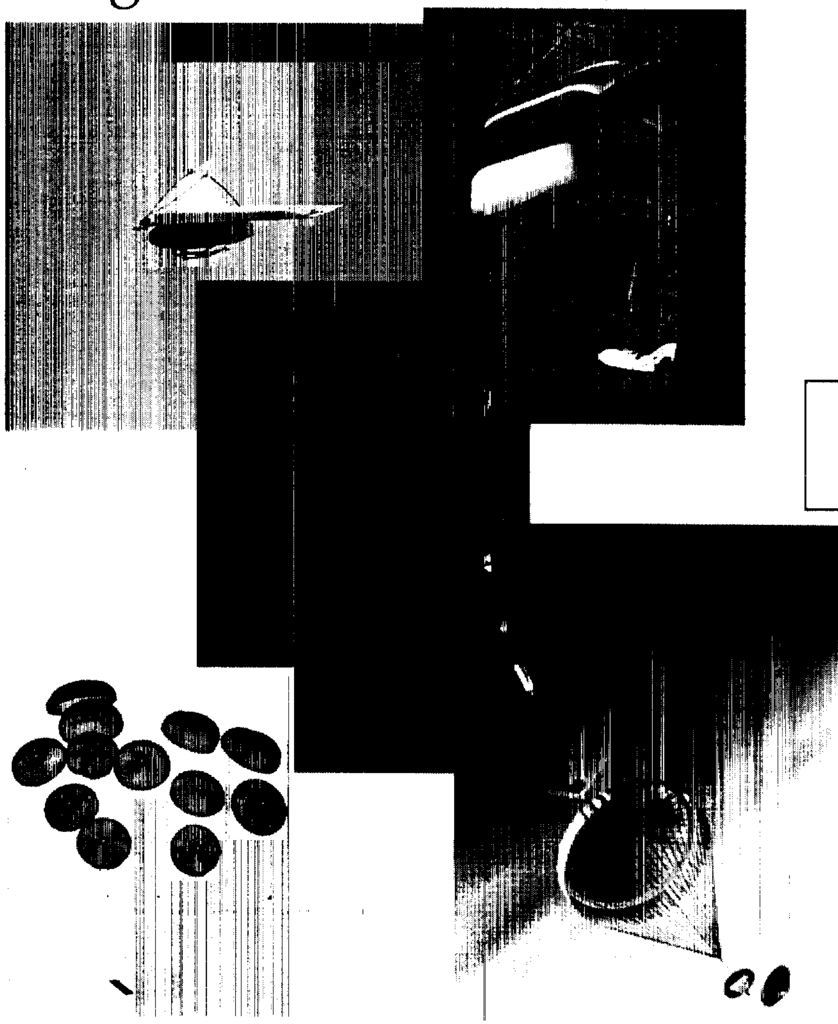
Measurement Technologies

Summary

R. Carpenter, 5/21/2001

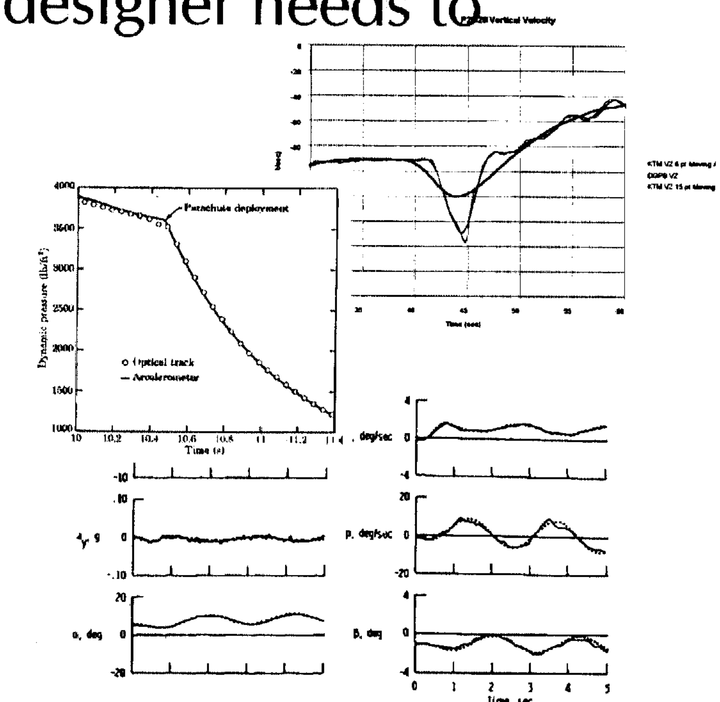
Problem Statement

Flight data sensors ...



... often don't measure, or don't measure accurately enough, what the designer needs to know:

?????



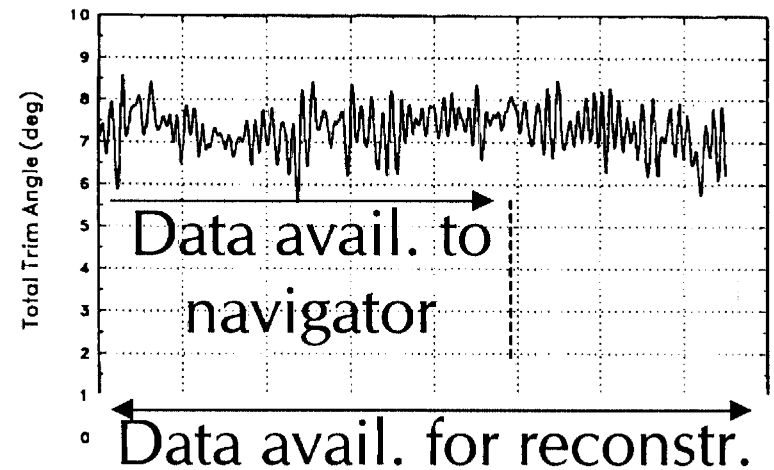
⇨ Analysts need to reconstruct engineering data from flight data

Goals

- Overview of sensors and methods for trajectory reconstruction
- Background for flight test designers
- Primer for beginning trajectory reconstruction analysts

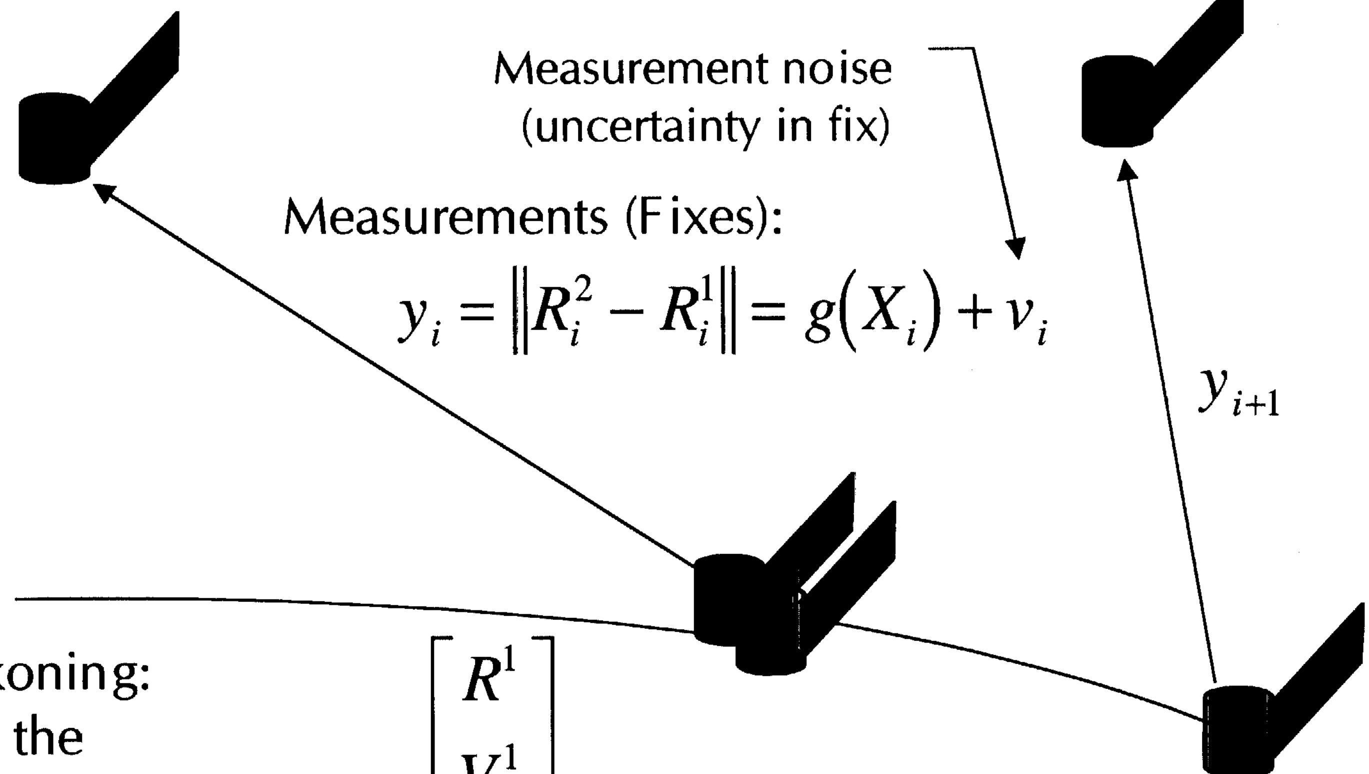
Overview

- Given:
  - Corrupt or incomplete data recorded during flight test
  - Models relating test data to data of interest
- Reconstruct as complete and accurate a record as possible of the data of interest



- Similar to navigation, except that at any point in the test, we know past, present, *and future*

The Navigation Process



Measurement noise
(uncertainty in fix)

Measurements (Fixes):

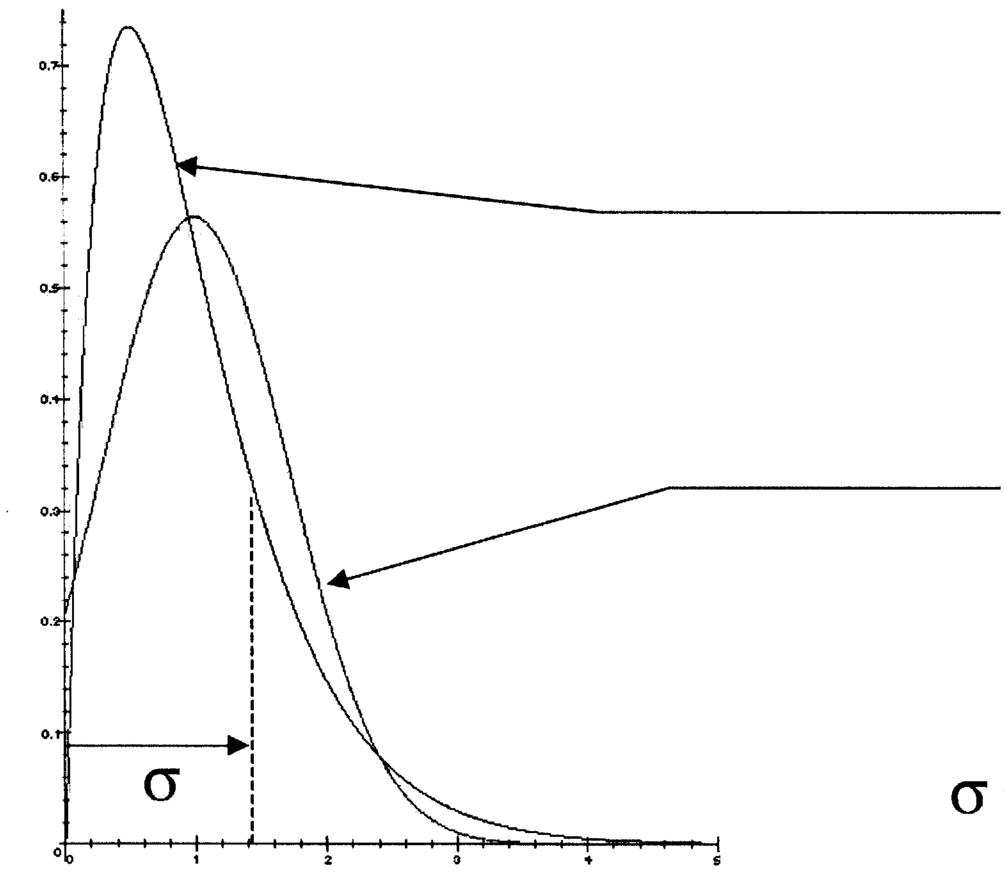
$$y_i = \left\| R_i^2 - R_i^1 \right\| = g(X_i) + v_i$$

Dead Reckoning:
Integrating the
Equations of Motion

$$\dot{\bar{X}} = \begin{bmatrix} R^1 \\ V^1 \\ R^2 \\ V^2 \end{bmatrix} = f(\bar{X}) + w$$

State noise
(uncert. in DR)

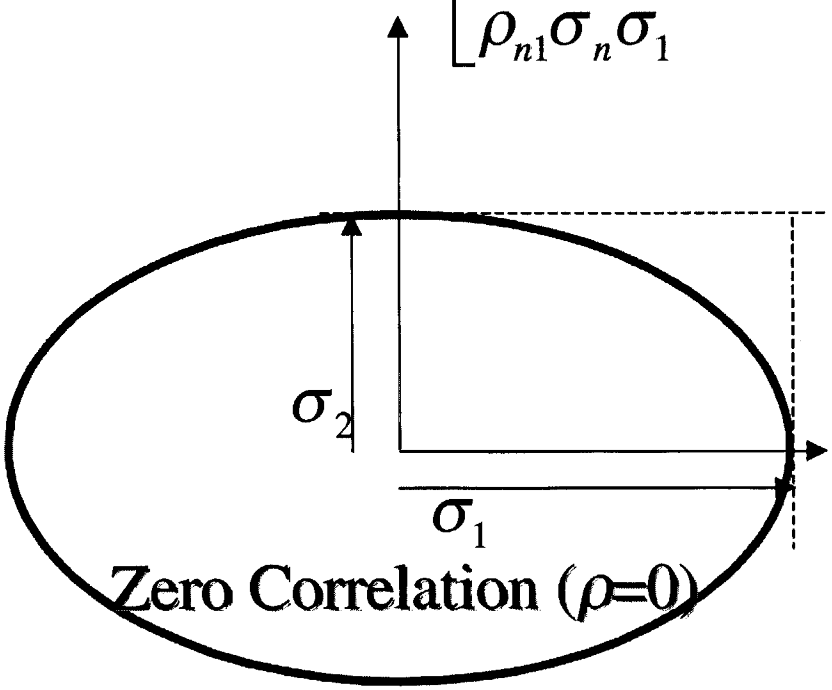
Information and Uncertainty



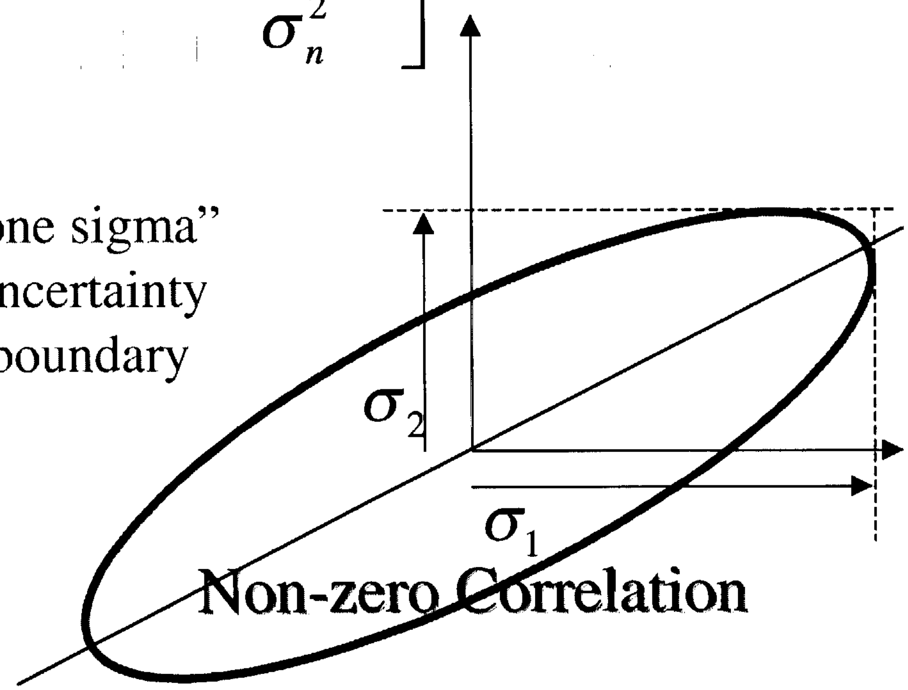
σ = standard deviation

Covariance Matrix

$$P = \begin{bmatrix} \sigma_1^2 & \rho_{12}\sigma_1\sigma_2 & \cdots & \rho_{1n}\sigma_1\sigma_n \\ \rho_{21}\sigma_2\sigma_1 & \sigma_2^2 & & \\ \vdots & & \ddots & \\ \rho_{n1}\sigma_n\sigma_1 & & & \sigma_n^2 \end{bmatrix}$$



"one sigma" uncertainty boundary



Navigation Filtering

$$\hat{X}(t_i) = \underbrace{K_1 \overline{X}(t_i)}_{\text{Dead Reckoning}} + \underbrace{K_2 Y(t_i)}_{\text{Fix}}$$

- Higher confidence in dead reckoning: choose $K1 > K2$
- Higher confidence = more information = less uncertainty = smaller covariance

Covariance Analysis

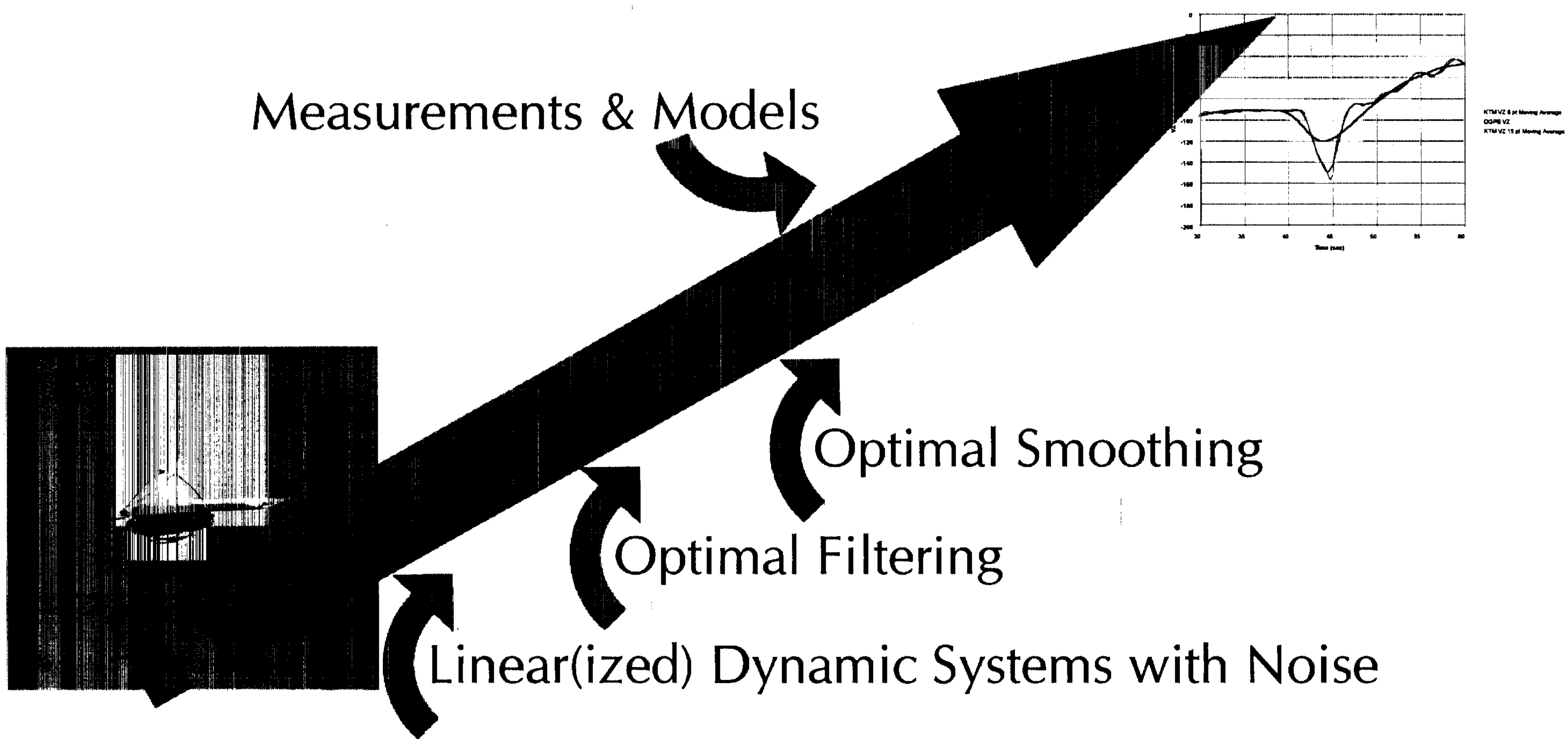
- Perform dead reckoning and fixing on covariance (details to come...)

$$\overline{P}(t_i) = \Phi(t_i, t_{i-1})\hat{P}(t_{i-1})\Phi^T(t_i, t_{i-1}) + Q(t_i)$$
$$\hat{P}^{-1}(t_i) = \overline{P}^{-1}(t_i) + H^T(t_i)R^{-1}(t_i)H(t_i)$$

  - Q = uncertainty about state noise
  - R = uncertainty about meas. noise
- Running total of uncertainty

⇨ Navigation filter does covariance analysis to pick *K1* and *K2*

Roadmap

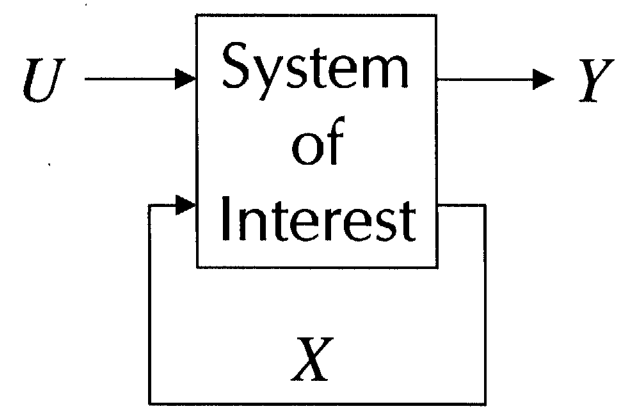

Measurements & Models
Optimal Smoothing
Optimal Filtering
Linear(ized) Dynamic Systems with Noise

Trajectory Reconstruction

- System Modeling
- Least squares & Batch Filtering
- Recursive Filtering
- Optimal Smoothing

System Model

- Model system of interest as a "black box"
  - Vector of inputs, U
  - Vector of outputs, Y
  - If inputs are not directly related to outputs, require additional internal parameters, X
- Model typically is nonlinear



$$Y(t_i) = g(t_i, X(t_i), U(t_i)) + v(t_i)$$

$$\dot{X}(t) = f(t, X(t), U(t)) + w(t)$$

State Vector

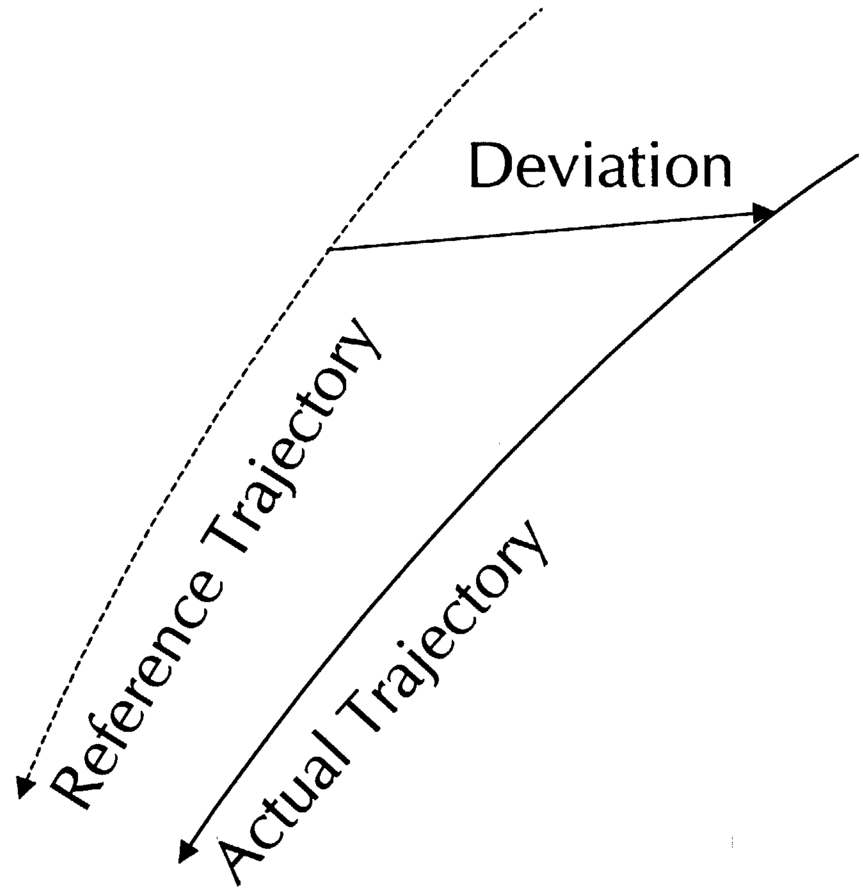
- The vector of internal parameters is called the "state vector"
- The state contains all the parameters one wants to reconstruct, and any other parameters needed to model the system, e.g. unknown calibration biases
- The state captures and compresses the history of the system; thus it is a representation of the system's "memory"

Observability

- With the available measurements, one may not be able to determine all of the parameters of interest independently
- Example: Ballistic coefficient, $m/Cd/A$, can be determined with GPS and air data, but the individual parameters, m, Cd, and A, cannot be
- "Observable" parameters are those that can be determined

Linearizing the System

- Nonlinear models are difficult to work with
- Suppose a "reference trajectory" exists, e.g. intended flight path
- Use Taylor's Theorem to express the actual trajectory in terms of deviations from the reference



Linearized System Model

$$\dot{X} - \dot{X}_{\mathrm{Ref}} = f\left(t, X_{\mathrm{Ref}} + x, U_{\mathrm{Ref}} + u\right) + w$$

$$= \left[\left.\frac{\partial f}{\partial X}\right|_{(X_{\mathrm{Ref}}, U_{\mathrm{Ref}})} \right] x + \left[\left.\frac{\partial f}{\partial U}\right|_{(X_{\mathrm{Ref}}, U_{\mathrm{Ref}})} \right] u + w$$

$$\boxed{\dot{x} = Ax + Bu + w}$$

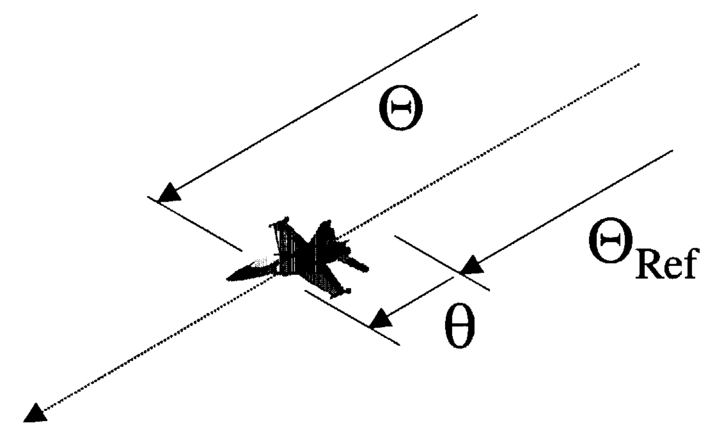
$$Y - Y_{\mathrm{Ref}} = g\left(t_i, X_{\mathrm{Ref}} + x, U_{\mathrm{Ref}} + u\right) + v$$

$$= \left[\left.\frac{\partial g}{\partial X}\right|_{(X_{\mathrm{Ref}}, U_{\mathrm{Ref}})} \right] x + \left[\left.\frac{\partial g}{\partial U}\right|_{(X_{\mathrm{Ref}}, U_{\mathrm{Ref}})} \right] u + v$$

$$\boxed{y = Cx + Du + v}$$

Example: Linearized 1-D Glider

- Consider along-track motion of glider with zero flight path angle



$$\dot{\Theta} = V$$

$$\dot{V} = -\frac{D}{M} = -\frac{\rho V^2}{2C_B}, \quad C_B = \frac{M}{C_D S}$$

$$\mathbf{X} = \begin{bmatrix} \Theta & V \end{bmatrix}^T$$

$$\dot{\mathbf{X}} = \mathbf{f}(\mathbf{X})$$

$$\begin{bmatrix} \dot{\Theta} \\ \dot{V} \end{bmatrix} = \begin{bmatrix} V & -\frac{\rho V^2}{2C_B} \end{bmatrix}^T$$

$$\dot{\mathbf{X}}_i - \mathbf{f}(\mathbf{X}_{i\mathrm{Ref}}) \approx \left[\left. \frac{\partial \mathbf{f}}{\partial \mathbf{X}} \right|_{\mathbf{X}_{\mathrm{Ref}}} \right] (\mathbf{X}_i - \mathbf{X}_{i\mathrm{Ref}})$$

$$= \begin{bmatrix} 0 & 1 \\ 0 & -\frac{\rho V_{i\mathrm{Ref}}}{C_B} \end{bmatrix} \begin{bmatrix} \theta_i \\ v_i \end{bmatrix}$$

$$\dot{\mathbf{x}}_i = A(V_{i\mathrm{Ref}})\mathbf{x}_i$$

Ex. Cont. - GPS, Air Data Meas.

- Suppose the glider has GPS and air data systems
- The GPS measures Θ, and the air data system measures qbar; hence

$$\mathbf{Y}_i = \left[\Theta_i \quad \bar{q}_i\right]^{\mathrm{T}}, \quad \bar{q}_i = \frac{\rho V_i^2}{2}$$

- Meas. Partials:

$$\mathbf{g}(\mathbf{X}) = \begin{bmatrix} \Theta & \dfrac{\rho V^2}{2} \end{bmatrix}^{\mathrm{T}}$$

$$\mathbf{Y}_i - \mathbf{g}\left(\mathbf{X}_{i\mathrm{Ref}}\right) \approx \left[\left.\frac{\partial \mathbf{g}}{\partial \mathbf{X}}\right|_{X_{\mathrm{Ref}}} \right] \left(\mathbf{X}_i - \mathbf{X}_{i\mathrm{Ref}}\right)$$

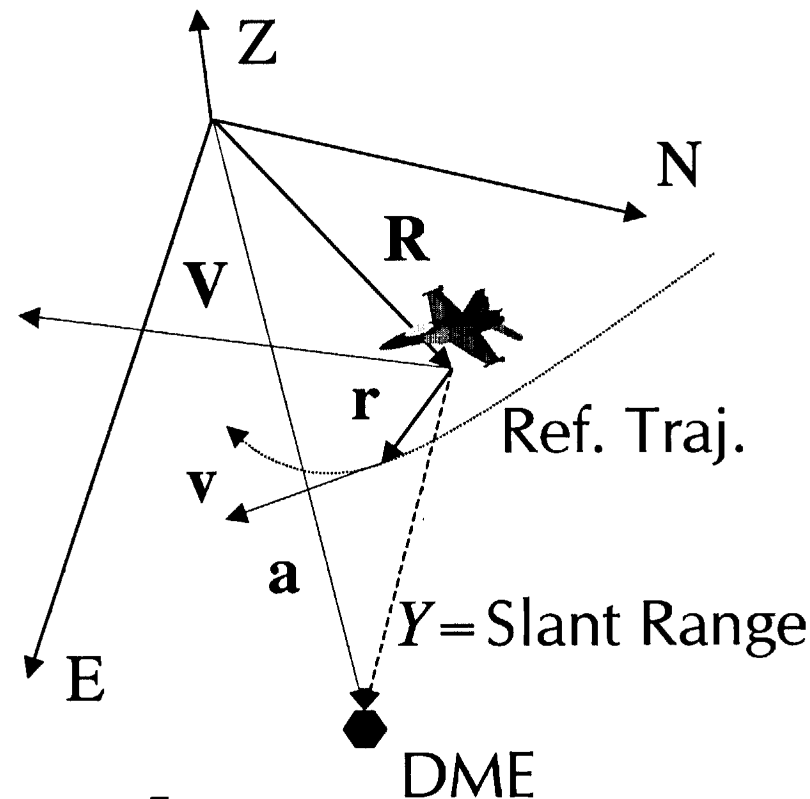
$$= \begin{bmatrix} 1 & 0 \\ 0 & \rho V_{i\mathrm{Ref}} \end{bmatrix} \begin{bmatrix} \theta_i \\ v_i \end{bmatrix}$$

$$\mathbf{y}_i = C\left(V_{i\mathrm{Ref}}\right)\mathbf{x}_i$$

Example: Linearized Range

- Distance Measuring Equipment (DME)
  - Slant range to station

$$Y_i = g(\mathbf{R}_i) + v_i$$
$$= \sqrt{(\mathbf{R}_i - \mathbf{a})^T (\mathbf{R}_i - \mathbf{a})} + v_i$$



- Linearized meas.:

$$Y_i - g(\mathbf{R}_{i\mathrm{Ref}}) \approx C[\mathbf{r}_i, \mathbf{v}_i]^{\mathrm{T}} + v_i$$

$$y_i = \left[\left. \frac{\partial \sqrt{(\mathbf{R}_i - \mathbf{a})^{\mathrm{T}} (\mathbf{R}_i - \mathbf{a})}}{\partial [\mathbf{R}_i, \mathbf{V}_i]^{\mathrm{T}}} \right|_{(\mathbf{R}_{i\mathrm{Ref}}, \mathbf{V}_{i\mathrm{Ref}})} \right] \mathbf{x}_i + v_i$$

Example: Range Meas. Contin.

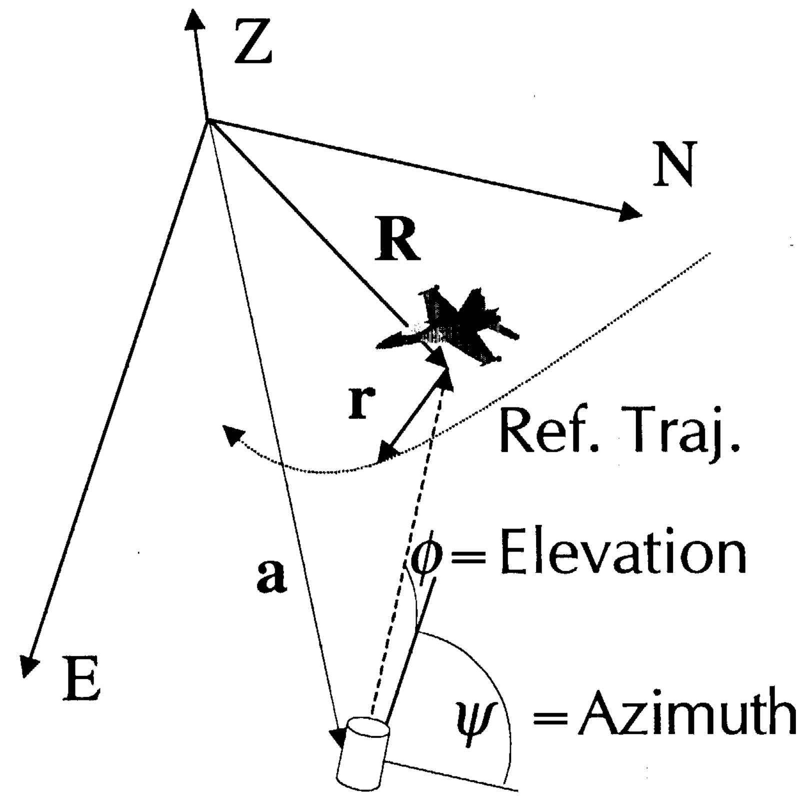
- Measurement Partial

$$C = \left[\frac{\partial \sqrt{(\mathbf{R}_i - \mathbf{a})^{\mathrm{T}} (\mathbf{R}_i - \mathbf{a})}}{\partial \mathbf{R}_i}, \quad \frac{\partial \sqrt{(\mathbf{R}_i - \mathbf{a})^{\mathrm{T}} (\mathbf{R}_i - \mathbf{a})}}{\partial \mathbf{V}_i} \right]_{\left(\mathbf{R}_{i\mathrm{Ref}}, \mathbf{V}_{i\mathrm{Ref}}\right)}$$

$$C\left(\mathbf{R}_{i\mathrm{Ref}}\right) = \left[\frac{\left(\mathbf{R}_{i\mathrm{Ref}} - \mathbf{a}\right)^{\mathrm{T}}}{\sqrt{\left(\mathbf{R}_{i\mathrm{Ref}} - \mathbf{a}\right)^{\mathrm{T}} \left(\mathbf{R}_{i\mathrm{Ref}} - \mathbf{a}\right)}} \quad 0 \quad 0 \quad 0 \right]$$

Example: Linearized Angles

- Optical tracking
- Camera measures Azimuth & Elevation to target



$$Az_i = g_{Az}(\mathbf{R}_i) + v_{Azi}$$

$$= \tan^{-1}\left(\frac{\mathbf{R}_{Ni} - \mathbf{a}_{N}}{\mathbf{R}_{Ei} - \mathbf{a}_{E}}\right) + v_{Azi}$$

$$El_i = g_{El}(\mathbf{R}_i) + v_{Eli}$$

$$= \tan^{-1}\left(\frac{-\mathbf{R}_{Zi} + \mathbf{a}_{Z}}{(\mathbf{R}_{Ei} - \mathbf{a}_{E})\cos Az_i + (\mathbf{R}_{Ni} - \mathbf{a}_{N})\sin Az_i}\right) + v_{Eli}$$

Example: Angle Meas. Contin.

- Linearized Meas.: $\mathbf{Y}_i - \mathbf{g}(\mathbf{R}_{i\mathrm{Ref}}) \approx C[\mathbf{r}_i, \mathbf{v}_i]^{\mathrm{T}} + \mathbf{v}_i$

$$\mathbf{y}_i = C(\mathbf{R}_{i\mathrm{Ref}})\mathbf{x}_i + \mathbf{v}_i$$

- Measurement Partial:

$$\Delta E = (\mathbf{R}_{\mathrm{E}i} - \mathbf{a}_{\mathrm{E}}), \quad \Delta N = (\mathbf{R}_{\mathrm{N}i} - \mathbf{a}_{\mathrm{N}}), \quad \Delta Z = (\mathbf{R}_{\mathrm{Z}i} - \mathbf{a}_{\mathrm{Z}})$$

$$D = \Delta E \cos Az + \Delta N \sin Az$$

$$C = \begin{bmatrix} \frac{\partial Az_i}{\partial \mathbf{R}_i} & \frac{\partial Az_i}{\partial \mathbf{V}_i} \\ \frac{\partial El_i}{\partial \mathbf{R}_i} & \frac{\partial El_i}{\partial \mathbf{V}_i} \end{bmatrix}_{(\mathbf{R}_{i\mathrm{Ref}}, \mathbf{V}_{i\mathrm{Ref}})}$$

$$= \begin{bmatrix} \left(\frac{-\Delta N}{\Delta E^2 + \Delta N^2}\right) & \left(\frac{\Delta E}{\Delta E^2 + \Delta N^2}\right) & 0 & \\ \left(\frac{\Delta Z \cos Az}{D^2 + \Delta Z^2}\right) & \left(\frac{\Delta Z \sin Az}{D^2 + \Delta Z^2}\right) & \left(\frac{-D^2}{D^2 + \Delta Z^2}\right) & O_{2\times 3} \end{bmatrix}$$

Discrete-time System Model

- Measurements are often not continuous, but occur in discrete samples
- The state deviations from the reference must be propagated between samples
- Let Δt = sample interval, then

$$x(t+\Delta t) = \Phi(t+\Delta t, t)x(t) + \int_t^{t+\Delta t} \Phi(\tau, t)B(\tau)u(\tau)d\tau$$

where Φ is the "state transition matrix"

State Transition Matrix

- In general, there is not a closed-form solution for Φ; must solve a matrix differential equation:

$$\frac{d\Phi(t,s)}{dt} = A(t)\Phi(t,s); \quad \Phi(s,s) = I$$

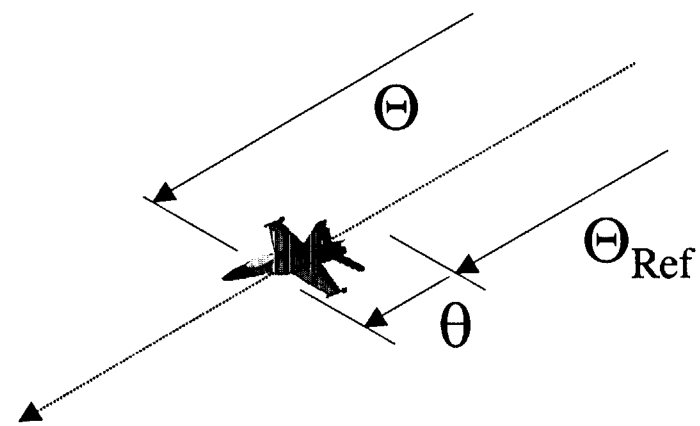
- If $A(t)$ is constant, or Δt "small enough"

$$\Phi(t + \Delta t, t) = \mathrm{e}^{A\Delta t} = I + A\Delta t + A^2 \frac{\Delta t^2}{2!} + \cdots$$

Example: STM for 1-D Glider

$$A_i = \begin{bmatrix} 0 & 1 \\ 0 & -\dfrac{\rho V_{i\mathrm{Ref}}}{C_B} \end{bmatrix}$$

$$\Phi(t_i + \Delta t, t_i) = \exp(A_i \Delta t) \approx I + A_i \Delta t$$



$$I + A_i \Delta t = \begin{bmatrix} 1 & \Delta t \\ 0 & 1 - \dfrac{\rho V_{i\mathrm{Ref}}}{C_B} \Delta t \end{bmatrix}$$

$$\exp(A_i \Delta t) = \begin{bmatrix} 1, & \dfrac{C_B}{\rho V_{i\mathrm{Ref}}}\left(1 - \exp\left(-\dfrac{\rho V_{i\mathrm{Ref}}}{C_B}\Delta t\right)\right) \\ 0, & \exp\left(-\dfrac{\rho V_{i\mathrm{Ref}}}{C_B}\Delta t\right) \end{bmatrix}$$

Estimating the State

- Find the state estimate that minimizes the weighted squared differences from the measured output
- Invertibility of $C'C$, implies, and is implied by, the observability of x
- "Least squares"

$$\tilde{y} = y - Du = Cx + v$$

$$\tilde{y} - C\hat{x}$$

$$J = \left(\tilde{y} - C\hat{x}\right)^T W \left(\tilde{y} - C\hat{x}\right)$$

$$\frac{\partial J}{\partial \hat{x}} = 0 \Rightarrow C^T W C \hat{x} = C^T W \tilde{y}$$

$$\hat{x} = \left(C^T W C\right)^{-1} C^T W \tilde{y}$$

Batch Filtering

- In general, the measurements don't occur all at the same time
- For dynamic systems, least squares must be modified to estimate the system state at the initial or other "anchor" time from a "batch" (time history) of measurements:

$$J = \sum_{i=1}^{N} \left(\tilde{y}_i - C\Phi(t_i,t_o)\hat{x}_o\right)^T W_i \left(\tilde{y}_i - C\Phi(t_i,t_o)\hat{x}_o\right)$$

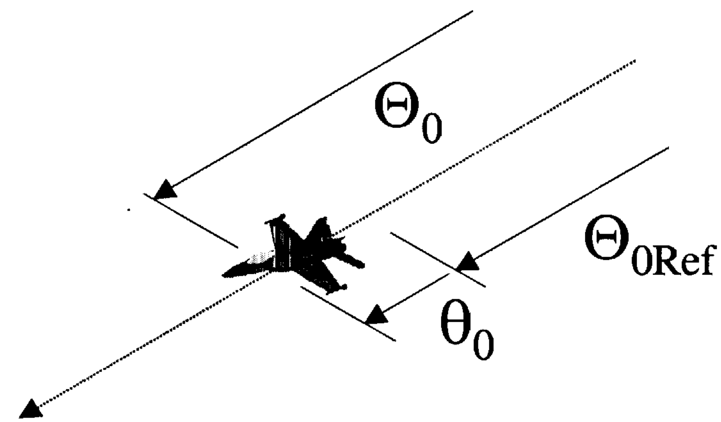
Batch Filtering, continued

- The state may be observable from the batch of measurements, even if it isn't from a single measurement epoch

$$\hat{x}_o = \left(\sum_{i=1}^{N} \Phi_i^T C_i^T W_i C_i \Phi_i \right)^{-1} \sum_{i=1}^{N} \Phi_i^T C_i^T W_i \tilde{y}_i$$

- Observability ⇔ invertibility $\Sigma\, \Phi' C' W C \Phi$
- The reference trajectory can be corrected with the newly estimated I.C., and the process iterated

Batch Filtering Example



- Problem: reconstruct I.C.'s & ballistic properties of 1-D glider
- Measurements:

$$Y_i = \begin{bmatrix} \Theta_i & \bar{q}_i \end{bmatrix}^T, \quad \bar{q}_i = \frac{\rho V_i^2}{2}, \quad i = 0,1$$

- Options for linearized state:
  - $x_1 = [\theta_0;\ v_0;\ c_B]$
  - $x_2 = [\theta_0;\ v_0;\ m;\ c_B;\ s]$
- Assume ballistic properties are constant
- *Check observability to decide which state vector to use*

Batch Filter Example Continued

$$X_1 = \begin{bmatrix} \Theta & V & C_B \end{bmatrix}^T$$

$$\dot{X}_1 = f(X)$$

$$\begin{bmatrix} \dot{\Theta} \\ \dot{V} \\ \dot{C}_B \end{bmatrix} = \begin{bmatrix} V & -\dfrac{\rho V^2}{2C_B} & 0 \end{bmatrix}^T$$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -\dfrac{\rho V_{iRef}}{C_{BRef}} & \dfrac{\rho V_{iRef}^{\;2}}{2C_{BRef}^{\;2}} \\ 0 & 0 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \rho V_{iRef} & 0 \end{bmatrix}$$

$$X_2 = \begin{bmatrix} \Theta & V & M & C_D & S \end{bmatrix}^T$$

$$\dot{X}_2 = f(X)$$

$$\begin{bmatrix} \dot{\Theta} & \dot{V} & \dot{M} & \dot{C}_D & \dot{S} \end{bmatrix}^T$$

$$= \begin{bmatrix} V & \dfrac{\rho V^2}{2C_B} & 0 & 0 & 0 \end{bmatrix}^T$$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & -\dfrac{\rho V_i C_D S}{M} & \dfrac{\rho V_i^2 C_D S}{2M^2} & -\dfrac{\rho V_i^2 S}{2M} & -\dfrac{\rho V_i^2 C_D}{2M} \\ O_{3\times1} & O_{3\times1} & O_{3\times1} & O_{3\times1} & O_{3\times1} \end{bmatrix}_{Ref}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \rho V_{iRef} & 0 & 0 & 0 \end{bmatrix}$$

Ex. Cont. - Observability Check

- $x_1 = [\theta_0; v_0; c_B]$
- $x_2 = [\theta_0; v_0; m; c_B; s]$

$$\Psi = \sum_{i=0}^{1} \Phi_i^T C_i^T C_i \Phi_i = C_0^T C_0 + \Phi_1^T C_1^T C_1 \Phi_1$$

- Ψ is a 3x3 matrix with rank 3
- x_1 is observable

- Ψ is a 5x5 matrix with rank 3
- x_2 is not observable

Batch Filter Example Concluded

- Given

$$Y_i = [\Theta_i \quad \bar{q}_i]^{\mathrm{T}}, \quad \bar{q}_i = \frac{\rho V_i^2}{2}, \quad i = 0,1$$

$$y_i = Y_i - g(X_{i\mathrm{Ref}})$$

- Solve for

$$\hat{x}_o = \left(C_0^T C_0 + \Phi_1^T C_1^T C_1 \Phi_1\right)^{-1}\left(C_0^T y_0 + \Phi_1^T C_1^T y_1\right)$$

- Use estimate to update reference values for $X_{\mathrm{Ref}} = [\Theta_{0\mathrm{Ref}};\ V_{0\mathrm{Ref}};\ C_{B\mathrm{Ref}}]$

Pros & Cons to Batch Filtering

- ✖ No accounting for disturbances (can use weighting matrix *W* to account for differences in measurement noise); dynamics model assumed to be perfect
- ✖ Only solves for "best fit" I.C. or anchor state; may not adequately characterize short-term trajectory deviations
- ✓ Fairly simple and easy to understand procedure

Looking for a better way

- What if there were a way to compute the mean and covariance that was
  - Recursive, i.e. used only information from the current and most recent sample times
  - Able to account for disturbances
  - Responsive to short-term variations
  - Based on statistics of measurements, disturbances, and initial condition uncertainties

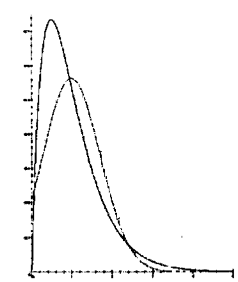
⇨ Kalman Filter

State Estimation Error

- Let e represent the state estimation error,

$$e = x - \hat{x}$$

- e is a random variable, i.e. it is a function that takes on different values based on the outcome of a random experiment
- The way in which the different values of e are distributed is governed by its probability density function, $p(e)$



Expected Value

- The mean of e is defined to be

$$m = \mathrm{E}[e(t)] = \int_{-\infty}^{\infty} e(t)p(e)de$$

- E[ ] is called the "expectation" or "expected value"
- The integral implies analysis of the "ensemble" of all possible random experiments
- In practice, we often use time averages

Covariance of the error

- The covariance is defined to be

$$P = \mathrm{E}\left[\{e(t) - \mathrm{E}[e(t)]\}\{e(t) - \mathrm{E}[e(t)]\}^{\mathrm{T}}\right]$$
$$= \int_{-\infty}^{\infty} \{e(t) - \mathrm{E}[e(t)]\}\{e(t) - \mathrm{E}[e(t)]\}^{\mathrm{T}} p(e)de$$

- As before, we often analyze a time history of a series of experiments, rather than explicitly perform integral above

Unbiased Recursive Filters

- Recall the filtering equation $$\hat{x} = K_1\bar{x} + K_2\tilde{y}$$
- The estimation error is $$\hat{e} = x - K_1(x + \bar{e}) - K_2(Cx + v)$$
- Assume that I.C.'s & disturbances are unbiased $$\mathrm{E}[\bar{x}] = 0 \Rightarrow \mathrm{E}[\bar{e}] = 0$$
- Choose K_1 to eliminate x $$K_1 = I - K_2C$$
- Then $$\mathrm{E}[\hat{e}] = \mathrm{E}\{[I - K_1 - K_2C]x\} - \mathrm{E}[K_1\bar{e}] + \mathrm{E}[K_2v] = 0$$

Unbiased Filter Covariance

- So $\hat{x} = [I - KC]\bar{x} + K\tilde{y}, \quad K = K_2$
- Its covariance is

$$\hat{P} = \mathrm{E}\left[\hat{e}\hat{e}^{\mathrm{T}}\right] = [I - KC]\mathrm{E}\left[\bar{e}\bar{e}^{\mathrm{T}}\right][I - KC]^{\mathrm{T}} + K\mathrm{E}\left[vv^{\mathrm{T}}\right]K^{\mathrm{T}}$$
$$+[I - KC]\mathrm{E}\left[\bar{e}v^{\mathrm{T}}\right]K^{\mathrm{T}} + K\mathrm{E}\left[v\bar{e}^{\mathrm{T}}\right][I - KC]^{\mathrm{T}}$$

- Let $\bar{P} = \mathrm{E}\left[\bar{e}\bar{e}^{\mathrm{T}}\right], \quad R = \mathrm{E}\left[vv^{\mathrm{T}}\right]$
- Assume $\mathrm{E}\left[\bar{e}v^{\mathrm{T}}\right] = \mathrm{E}\left[v\bar{e}^{\mathrm{T}}\right] = 0$
- Then $\hat{P} = [I - KC]\bar{P}[I - KC]^{\mathrm{T}} + KRK^{\mathrm{T}}$

Optimal Filter Gain

- Choose K by minimizing

$$\mathrm{J} = \mathrm{trace}\left(\hat{P}\right) = \mathrm{tr}\left([I - KC]\bar{P}[I - KC]^{\mathrm{T}} + KRK^{\mathrm{T}}\right)$$

- Use

$$\frac{\partial}{\partial A}\mathrm{trace}\left(ABA^{\mathrm{T}}\right) = 2AB$$

- To find

$$\left.\frac{\partial J}{\partial K}\right|_{K=K_{\mathrm{opt}}} = 0 = -2[I - KC]\bar{P}C^{\mathrm{T}} + 2KR$$

$$\Rightarrow K_{\mathrm{opt}} = \bar{P}C^{\mathrm{T}}\left(C\bar{P}C^{\mathrm{T}} + R\right)^{-1}$$

Propagation

- Between measurements,

$$\bar{x}_i = \Phi(t_i, t_{i-1})\hat{x}_{i-1} + \int_{t_{i-1}}^{t_i} \Phi(\tau, t)B(\tau)u(\tau)d\tau$$

$$\begin{aligned}
\bar{P}_i &= \mathrm{E}\left[\bar{e}_{i-1}\bar{e}_{i-1}^{\mathrm{T}}\right] \\
&= \mathrm{E}\left\{\left[\Phi(t_i, t_{i-1})\hat{e}_{i-1} + w_{i-1}\right]\left[\Phi(t_i, t_{i-1})\hat{e}_{i-1} + w_{i-1}\right]^{\mathrm{T}}\right\} \\
&= \Phi(t_i, t_{i-1})\mathrm{E}\left[\hat{e}_{i-1}\hat{e}_{i-1}^{\mathrm{T}}\right]\Phi(t_i, t_{i-1})^{\mathrm{T}} + \mathrm{E}\left[w_{i-1}w_{i-1}^{\mathrm{T}}\right] \\
&\quad + \Phi(t_i, t_{i-1})\mathrm{E}\left[\hat{e}_{i-1}w_{i-1}^{\mathrm{T}}\right] + \mathrm{E}\left[w_{i-1}\hat{e}_{i-1}^{\mathrm{T}}\right]\Phi(t_i, t_{i-1})^{\mathrm{T}} \\
&= \Phi(t_i, t_{i-1})\hat{P}_{i-1}\Phi(t_i, t_{i-1})^{\mathrm{T}} + Q_{i-1}
\end{aligned}$$

Discrete Kalman Filter

- In summary, the recursive filter is

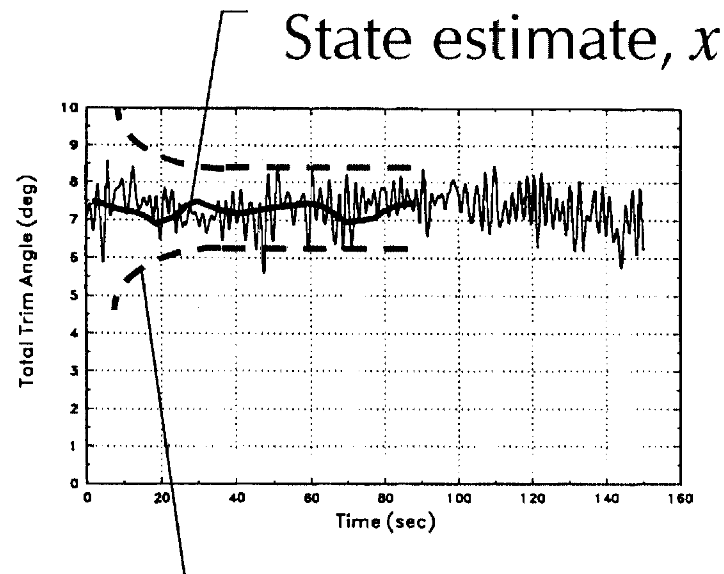
$$\bar{x}_i = \Phi(t_i, t_{i-1})\hat{x}_{i-1} + \int_{t_{i-1}}^{t_i} \Phi(\tau, t)B(\tau)u(\tau)d\tau$$

$$\bar{P}_i = \Phi(t_i, t_{i-1})\hat{P}_{i-1}\Phi(t_i, t_{i-1})^{\mathrm{T}} + Q_{i-1}$$

$$K_i = \bar{P}_i C_i^{\mathrm{T}}\left(C_i \bar{P}_i C_i^{\mathrm{T}} + R_i\right)^{-1}$$

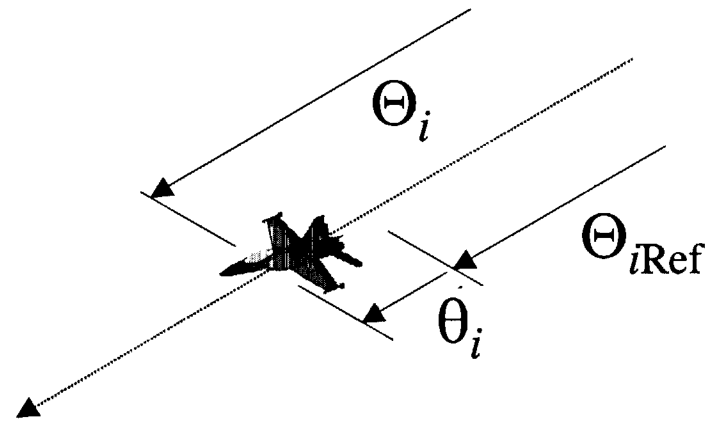
$$\hat{x}_i = [I - K_i C_i]\bar{x}_i + K_i \tilde{y}_i$$

$$\hat{P}_i = [I - K_i C_i]\bar{P}_i[I - K_i C_i]^{\mathrm{T}} + K_i R_i K_i^{\mathrm{T}}$$



Kalman Filter Example

- Problem: reconstruct trajectory & ballistic properties of 1-D glider



- Measurements:

$$Y_i = \left[\Theta_i + n_{\Theta i} \quad \bar{q}_i + n_{qi}\right]^{\mathrm{T}}, \quad \bar{q}_i = \frac{\rho V_i^2}{2}, \quad i = 0,1,\ldots,N$$

- Linearized state:

$$x_i = [\theta_i;\ v_i;\ c_B]$$

- Initial state $=$ $\mathrm{E}[x_0] = 0$

- Initial covariance

$$P_0 = \begin{bmatrix} \mathrm{E}[\theta_0^2] & 0 & 0 \\ 0 & \mathrm{E}[v_0^2] & 0 \\ 0 & 0 & \mathrm{E}[c_{B0}^2] \end{bmatrix}$$

Kalman Filter Example Contin.

- Measurement noise covariance is nominally based on the sensor noise variances

$$R_i = \begin{bmatrix} \mathrm{E}\left[n_{i\Theta}^2\right] & 0 \\ 0 & \mathrm{E}\left[n_{iq}^2\right] \end{bmatrix}$$

- Process noise covariance is nominally based on the variances of the disturbances

$$Q_i = \begin{bmatrix} \mathrm{E}\left[w_{\theta i}^2\right] & 0 & 0 \\ 0 & \mathrm{E}\left[w_{qi}^2\right] & 0 \\ 0 & 0 & \mathrm{E}\left[w_{c_B i}^2\right] \end{bmatrix}$$

- R & Q are often "tuned" or manually adjusted to compensate for other approx.

Kalman Filter Example Concl.

- As each measurement arrives ...

$$Y_i = [\Theta_i \quad \bar{q}_i]^{\mathrm{T}}, \quad \bar{q}_i = \frac{\rho V_i^2}{2}, \quad i = 0,1,\ldots,N$$

$$y_i = Y_i - g(X_{i\mathrm{Ref}})$$

- Update the state:

$$\hat{x}_i = [I - K_i C_i]\bar{x}_i + K_i y_i$$

- Use estimate to update reference values for $X_{i\mathrm{Ref}} = [\Theta_{i\mathrm{Ref}};\ V_{i\mathrm{Ref}};\ C_{i\mathrm{Ref}}]$

Continuous Kalman Filter

- Suppose we have analog sensors
- Measurements are continuously available
- An analog computer could continuously process data
- Could we use the Kalman Filter? Yes!

- Letting $\Delta t \to 0$, it can be shown (e.g. Brown & Hwang, Ch. 7) that the continuous-time form of the Kalman filter is

$$\dot{\hat{x}} = A\hat{x} + Bu + K(y - C\hat{x})$$

$$K = \hat{P}C^{\mathrm{T}}R^{-1}$$

$$\dot{\hat{P}} = A\hat{P} + \hat{P}A^{\mathrm{T}} - \hat{P}C^{\mathrm{T}}R^{-1}C\hat{P} + Q$$

Cont./Discrete Kalman Filter

- Most physical systems are modeled by continuous differential equations
- Most sensors take discrete measurements
- Most computers are digital, but they can solve continuous d.e.'s
- We'd like to use a continuous model between discrete measurements
- Therefore, the most common form of the Kalman filter is a hybrid of the discrete and continuous forms

Cont./Disc. K.F. Recursion

- Between measurements, integrate

$$\dot{\bar{x}}(t) = A(t)\bar{x}(t) + B(t)u(t), \quad \bar{x}(t_{i-1}) = \hat{x}_{i-1}$$

$$\dot{\Phi}(t, t_{i-1}) = A(t)\Phi(t, t_{i-1}), \quad \Phi(t_{i-1}, t_{i-1}) = I$$

$$\bar{P}_i = \Phi(t_i, t_{i-1})\hat{P}_{i-1}\Phi(t_i, t_{i-1})^{\mathrm{T}} + Q_{i-1}$$

(Numerical integration of Pdot is often not accurate, since d.e. is "stiff")

- At measurement times, update

$$K_i = \bar{P}_i C_i^{\mathrm{T}} \left(C_i \bar{P}_i C_i^{\mathrm{T}} + R_i\right)^{-1}$$

$$\hat{x}_i = [I - K_i C_i]\bar{x}_i + K_i \tilde{y}_i$$

$$\hat{P}_i = [I - K_i C_i]\bar{P}_i[I - K_i C_i]^{\mathrm{T}} + K_i R_i K_i^{\mathrm{T}}$$

Nonlinear Filtering

- Like the batch filter, the Kalman filter operates on linearized deviations from the reference trajectory
- We'd like to operate directly on the full nonlinear state and measurement
- Nonlinear filtering is an advanced topic, and generally requires solution of partial differential equations defining the probability density function
- But, an ad hoc procedure based on the K.F. exists...

Extended Kalman Filter

- Between measurements, integrate

$$\dot{\overline{X}}(t) = f\left(t, \overline{X}, U\right), \quad \overline{X}\left(t_{i-1}\right) = \hat{X}_{i-1}$$

$$\dot{\Phi}\left(t, t_{i-1}\right) = A(t)\Phi\left(t, t_{i-1}\right), \quad \Phi\left(t_{i-1}, t_{i-1}\right) = I$$

$$\overline{P}_i = \Phi\left(t_i, t_{i-1}\right)\hat{P}_{i-1}\Phi\left(t_i, t_{i-1}\right)^{\mathrm{T}} + Q_{i-1}$$

- At measurement times, update

$$K_i = \overline{P}_i C_i^{\mathrm{T}}\left(C_i \overline{P}_i C_i^{\mathrm{T}} + R_i\right)^{-1}$$

$$\hat{X}_i = \overline{X}_i + K_i\left[Y_i - g\left(t_i, \overline{X}_i, U_i\right)\right]$$

$$\hat{P}_i = \left[I - K_i C_i\right]\overline{P}_i\left[I - K_i C_i\right]^{\mathrm{T}} + K_i R_i K_i^{\mathrm{T}}$$

Suboptimal Filter Design

- Nearly all practical filters are suboptimal, due to various approximations
  - Linearization, inaccurate parameters, truncated models, etc.
- Achieving accuracy, stability, and robustness with a suboptimal filter is a time-consuming and challenging process
- Navigation remains an "art," not a science...

Correlated Errors

- In deriving the Kalman filter, we (implicitly) assumed that the process and measurement noise are not correlated
- If they are correlated, the gain and covariance update must be modified

- Assume

$$S = \mathrm{E}\left[vw^{\mathrm{T}}\right], \quad S^{\mathrm{T}} = \mathrm{E}\left[wv^{\mathrm{T}}\right]$$

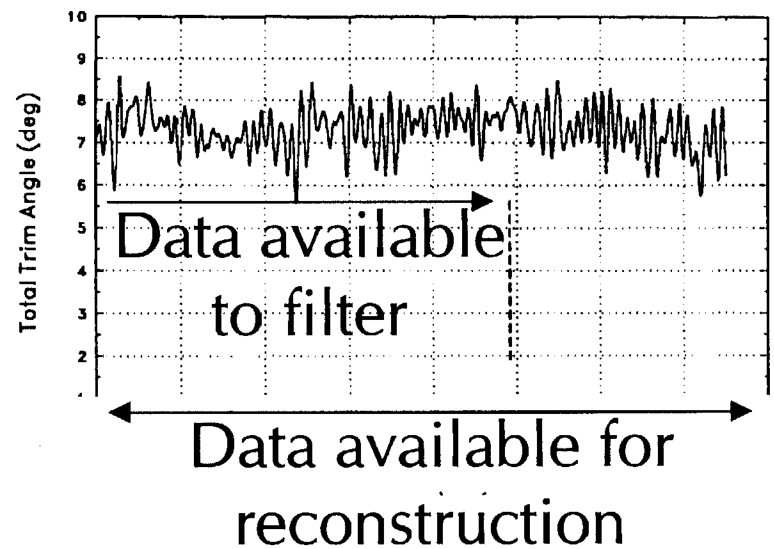
- Then

$$K_i = \left(\overline{P}_i C_i^{\mathrm{T}} + S_i\right) \cdot \left(C_i \overline{P}_i C_i^{\mathrm{T}} + R_i + C_i S_i + S_i^{\mathrm{T}} C_i^{\mathrm{T}}\right)^{-1}$$

$$\hat{P}_i = \left[I - K_i C_i\right]\overline{P}_i\left[I - K_i C_i\right]^{\mathrm{T}} + K_i R_i K_i^{\mathrm{T}} - \left[I - K_i C_i\right] S_i K_i^{\mathrm{T}} - K_i C_i^{\mathrm{T}}\left[I - K_i C_i\right]^{\mathrm{T}}$$

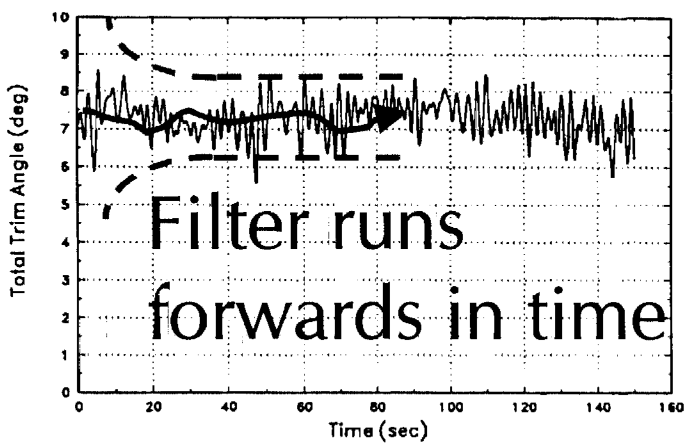
Using Future Data

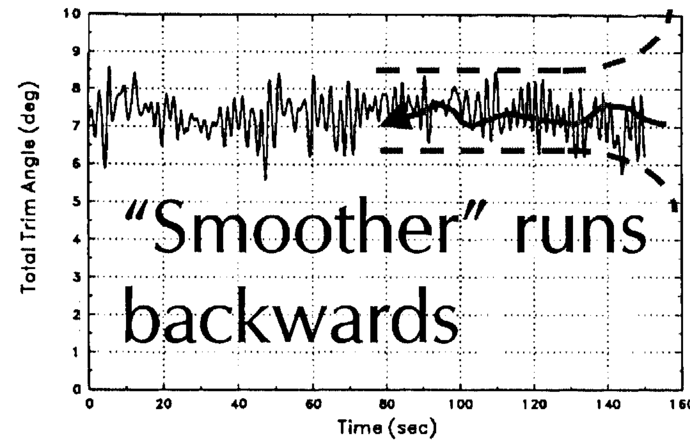
- Filtering is a real-time process
- Access is to future information is denied
- After the fact, one has the entire time history of the data
- For trajectory reconstruction, want to use *all* the data



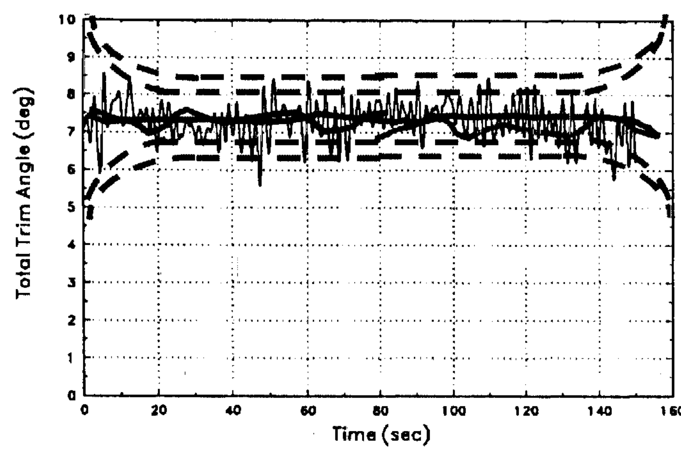
Filter/Smoothers

- To use all the data, run 2 filters





"Fuse" the two estimates at the intermediate times...



...to get the best estimate using all the data

Unbiased Filter/Smoother

- Fusion of filter & smoother
$$\hat{x} = K_1\hat{x}_F + K_2\hat{x}_S$$
- The estimation error is
$$\hat{e} = x - K_1\left(x + \hat{e}_F\right) - K_2\left(x + \hat{e}_2\right)$$
- Assume filter & smoother unbiased
$$\mathrm{E}\left[\hat{x}_F\right] = 0 \Rightarrow \mathrm{E}\left[\hat{e}_F\right] = 0$$
$$\mathrm{E}\left[\hat{x}_S\right] = 0 \Rightarrow \mathrm{E}\left[\hat{e}_S\right] = 0$$
- Choose *K1* to eliminate x
$$K_1 = I - K_2$$
- Then $\mathrm{E}[\hat{e}] = \mathrm{E}\left\{\left[I - K_1 - K_2\right]x\right\} + \mathrm{E}\left[K_1\hat{e}_F\right] + \mathrm{E}\left[K_2\hat{e}_S\right] = 0$
- Covariance is $\hat{P} = K\hat{P}_F K^{\mathrm{T}} + [I - K]\hat{P}_S[I - K]^{\mathrm{T}}$

Optimal Filter/Smoother

- Choose K by minimizing

$$J = \mathrm{trace}\left(\hat{P}\right) = \mathrm{tr}\left(K\hat{P}_F K^{\mathrm{T}} + [I-K]\hat{P}_S[I-K]^{\mathrm{T}}\right)$$

- To find

$$\left.\frac{\partial J}{\partial K}\right|_{K=K_{\mathrm{opt}}} = 0 = 2K\hat{P}_F + 2[I-K]\hat{P}_S$$

$$\Rightarrow K_{\mathrm{opt}} = \hat{P}_S\left(\hat{P}_F + \hat{P}_B\right)^{-1}, \quad I - K_{\mathrm{opt}} = \hat{P}_F\left(\hat{P}_F + \hat{P}_B\right)^{-1}$$

- With K_{opt}, the fused covariance becomes

$$\hat{P} = \left(\hat{P}_F^{-1} + \hat{P}_S^{-1}\right)^{-1}$$

- … and the fused state becomes

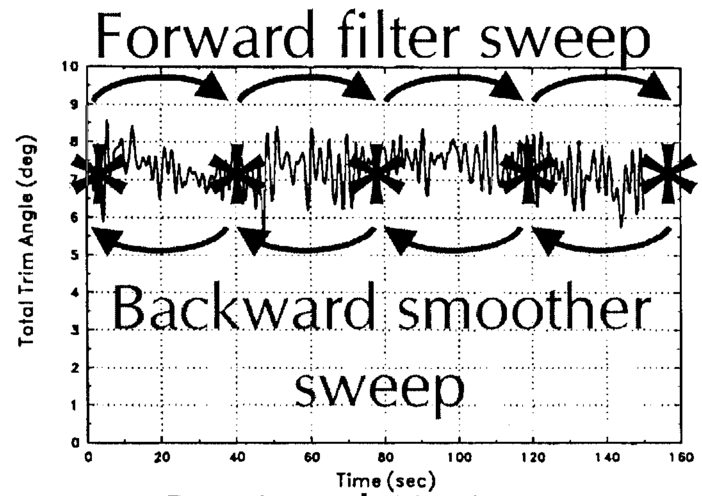
$$\hat{x} = \hat{P}\left(\hat{P}_F^{-1}\hat{x}_F + \hat{P}_S^{-1}\hat{x}_S\right)$$

Smoothability

- If filter/smoother estimate of a state is more accurate than filter estimate alone, state is said to be *smoothable*.
- Smoothable state are those that are controllable with respect to the process noise
  - Constant states, which have no process noise, are not smoothable
  - Randomly time-varying states are smoothable

Fixed-interval Smoothing

- The type of smoother just discussed is a "fixed-interval" smoother
  - Measurement data interval is fixed
  - Optimal estimates are to be found at interior points
  - Most typical for post-flight trajectory reconstruction

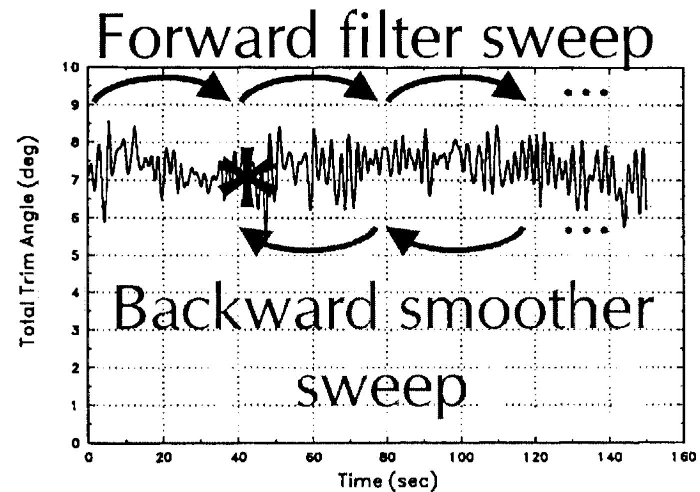


* Optimal Estimates

- Other types of smoothers useful for open-ended data sets:
  - Fixed-point
  - Fixed-lag

Fixed-lag Smoothing

- Suppose data continues indefinitely into the future
- As each measurement is processed, perform a short-term smoother sweep backwards a few steps

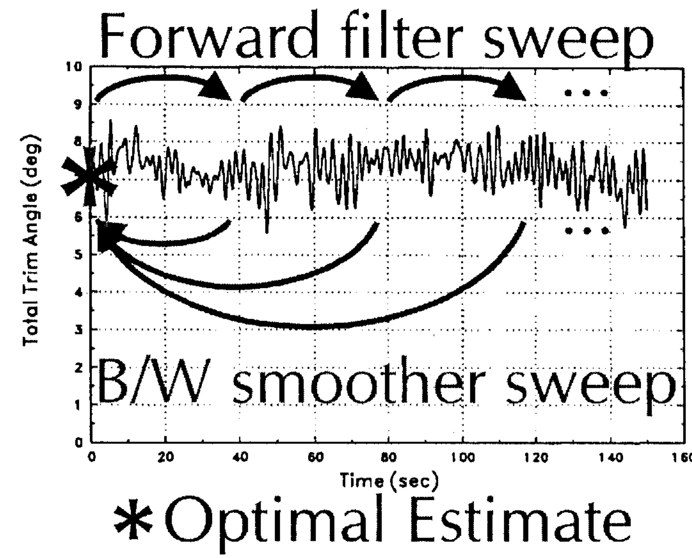


*Optimal Estimate

- Result is an optimal estimate that lags real-time by a few steps

Fixed-point Smoothing

- Suppose one only seeks an optimal estimate a a fixed point in time
  - Estimating I.C.'s
- As each new measurement is processed, perform a smoother sweep all the way back to the fixed point



- Often find that data processed far into the fixed point's future has little effect

Smoothing Comments

- Many algorithms exist that accomplish the types of smoothing just described
- Ch. 8 in Brown & Hwang and Ch. 5 in Gelb contain overviews and descriptions of many of these

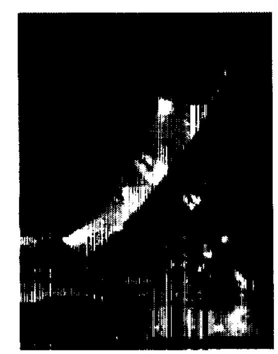
Measurement Technologies

- Ground tracking systems
- Onboard navigation systems

Ground Tracking Data

- Optical tracking systems
  - Cameras provide relative azimuth and elevation
  - Options include IR for night sensing, laser ranging
  - Often small & easily transportable

- LASER & RADAR tracking
  - Typically fixed stations
  - Range, range-rate, az, el

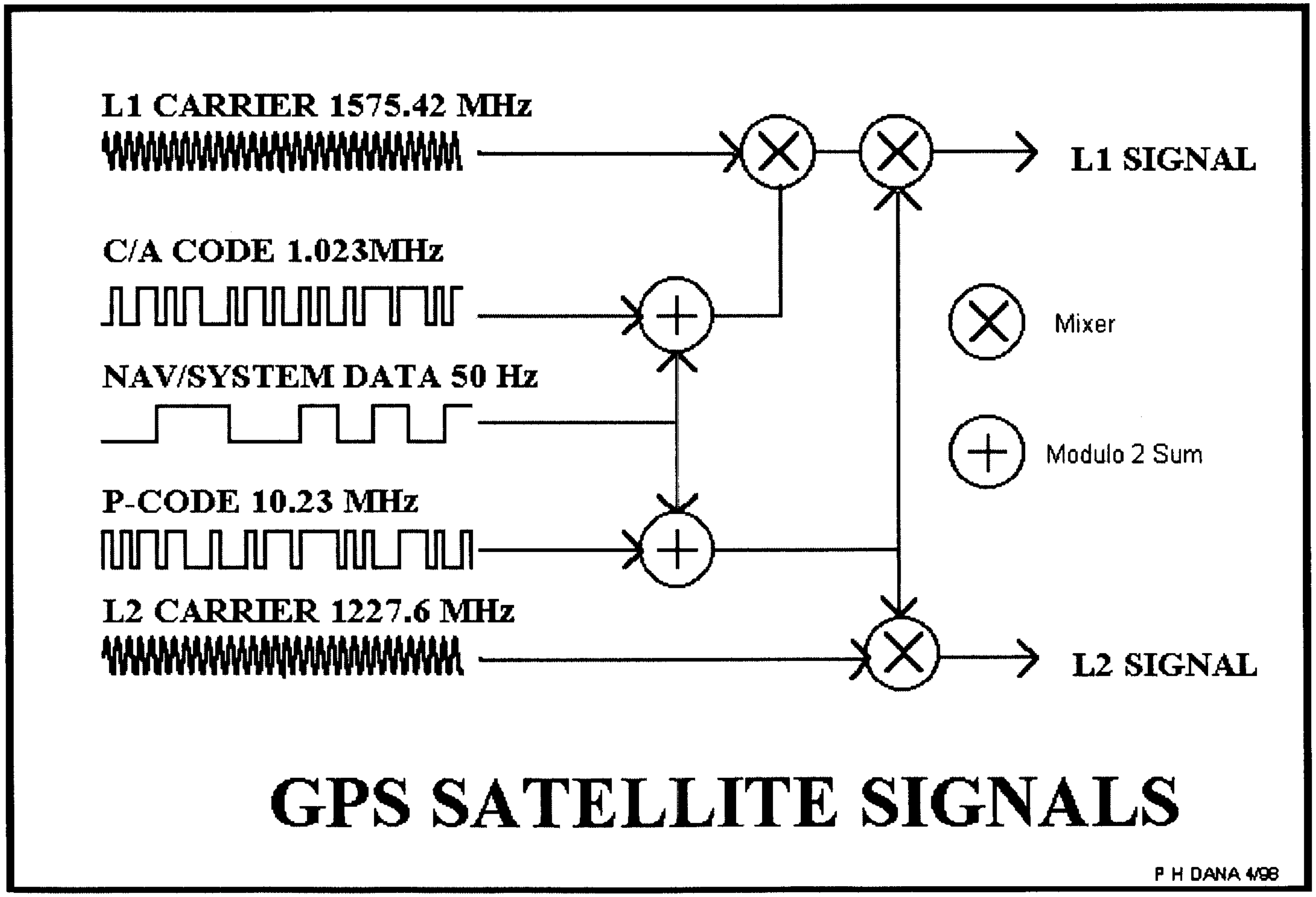


Global Positioning System

- 21 satellites + 3 spares
- 6 planes with 4 satellites each
- 12-hour periods
- 55-degree inclinations

- GPS receivers measure ranges to 4 or more satellites to determine position and time

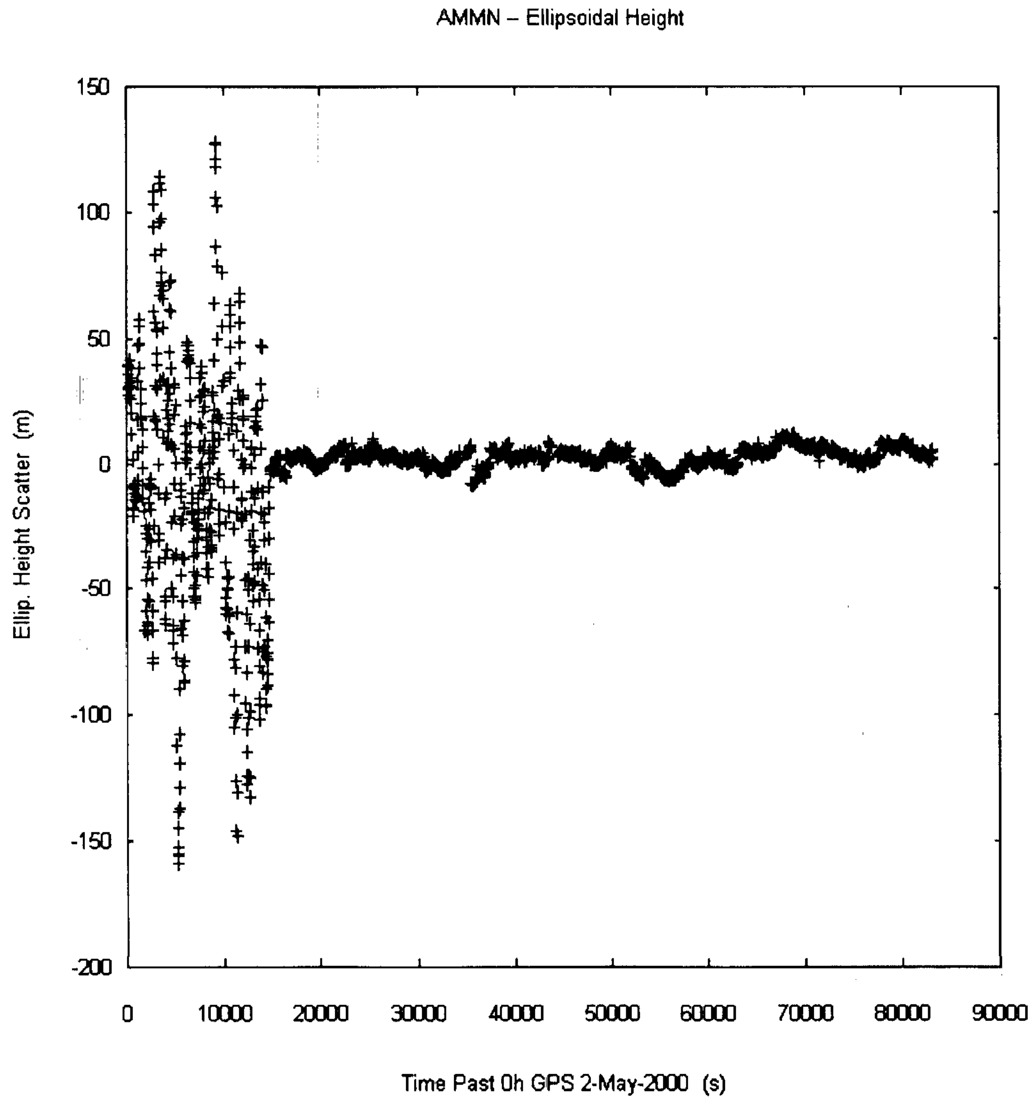
GPS Signal Structure



Standard Positioning Service

- Non-authorized users have access only to the L1 carrier
- SPS receiver operation:
  - Demodulate the C/A code and system message from the received carrier
  - Generate a local copy of the C/A code
  - Autocorrelate the local code with the received code at various time shifts
  - The signals correlate when the time shift
    = signal transit time
    = range * spd of light
- Sys. msg ⇨ GPS pos.
- 3 ranges ⇨ position
- 4th range ⇨ time

Selective Availability



- Prior to May 2, 2000, the DoD intentionally degraded the GPS SPS signals
- Selective availability has been replaced by "selective denial"

GPS SPS Measurement Model

- Let $\mathbf{R}\_SVj$ = position of jth GPS satellite, b_Svj = jth GPS range bias
- Let $\mathbf{R}$ = user position, Δt = user clock bias
- The range measurement is

$$y_j = \left\| \mathbf{R} - \mathbf{R}_{SVj} \right\| + c\Delta t + b_{SVj} + v_j$$

- Use least squares to solve for $\mathbf{R}$ and Δt with ≥4 ranges; called a "point solution"

Other Measurement Types

- Precise Positioning Service - correlate encrypted code on L2 carrier
  - Must be "authorized users" to decrypt
  - Use of 2nd frequency allows cancellation of atmospheric & ionospheric refraction biases
- Ranging using the carrier phase
- Differential GPS/GPS interferometry
- GPS Modernization: additional civil signals

Carrier Phase Data

- To track the carrier, GPS receivers must match changes in its phase and/or freq.
- The wavelength of the L1 carrier is 19 cm; phase tracking precision is ~mm
- Thus, carrier phase is a very precise range measurement
- However, the receiver only measures fractional phase (modulo 2π) at acquisition, then subsequent phase changes
- The initial integer number of phase cycles is unknown

Carrier Ambiguity Resolution

- The ambiguity is typically found via search methods
- A popular technique (LAMBDA) uses transformations of the search space that decorrelate the ambiguities*
- In surveying applications, the initial integer phase cycle ambiguity is routinely found
- For dynamic applications, ambiguity resolution is becoming more reliable

*See http://www.geo.tudelft.nl/mgp/lambda/index.html

Differential GPS

- GPS receiver at precisely surveyed station
- Determines errors in GPS signal (b_Svj = jth GPS range bias)
- Broadcasts error data to nearby users (within ~100 miles)
- USCG has DGPS base stations that cover both coasts and the Great Lakes
- A separate receiver is required to receive the DGPS broadcast

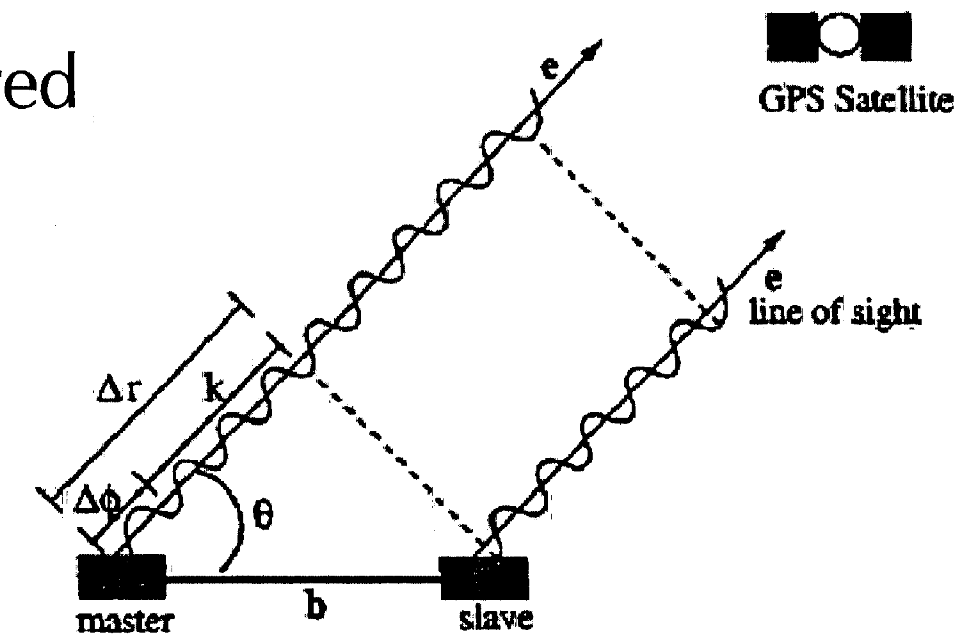
Single & Double Differences

- Similarly, user can difference base station's data with their own, to remove the bias
  - "Receiver-to-receiver single difference"
  - Differencing increases the noise by $\sqrt{2}$
- A second difference between GPS SVs can be made to remove the clock bias
  - "Receiver/satellite double difference"
  - Increases the noise by another factor $\sqrt{2}$

The International GPS Service, http://igscb.jpl.nasa.gov/, maintains GPS measurement data from a worldwide network of base stations

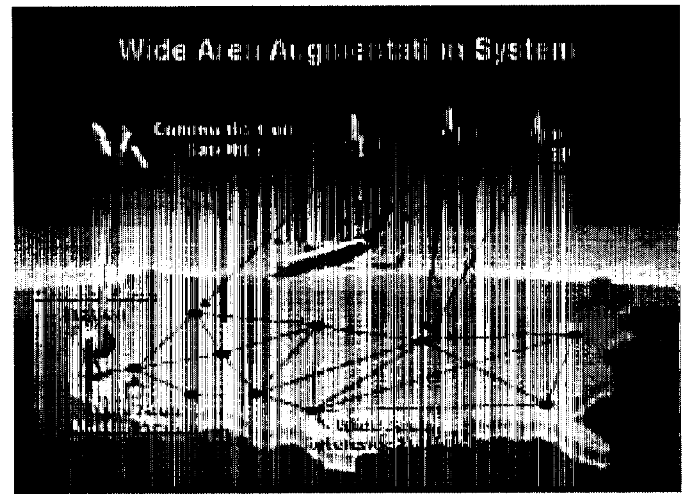
GPS Attitude Determination

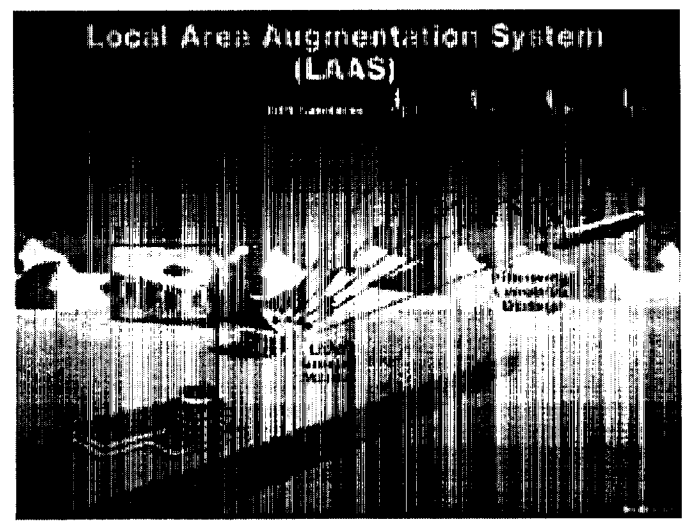
- Carrier-phase DGPS interferometry
- 3-4 antennas required
- Difficulties:
  - Differential cycle ambiguity & cycle slips
  - Multipath
- To be used by International Space Station



GPS "Modernization"

- WAAS - Cat 1 approaches
- LAAS - Cat 2/3 approaches
- Additional civil frequencies:
  - C/A on L2: 1st satellite 2003
  - L5 (1176.45 MHz): 1st launch 2004-5.
- Full capability: 2010



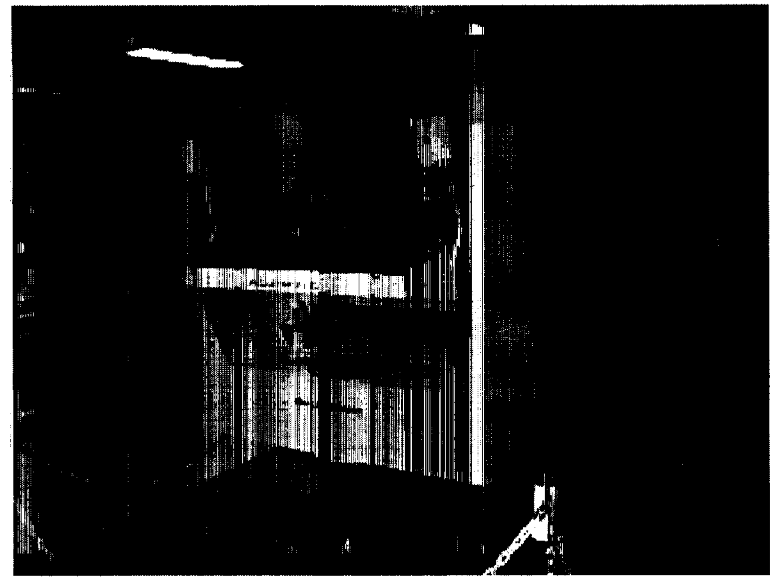


Other SatNav systems

- Current generic term is GNSS: Global Navigation Satellite Systems
- GLONASS
  - Former Soviet version of GPS; future?
  - http://www.rssi.ru/SFCSIC/english.html
- GALILEO
  - European Union proposed
  - http://www.galileo-pgm.org/
- Interoperability is an issue of contention...

Inertial Navigation Systems

- All INS systems maintain the orientation of a set of accelerometers and gyros (platform)
- Motions of the vehicle around the platform are sensed by the gyros and accelerometers

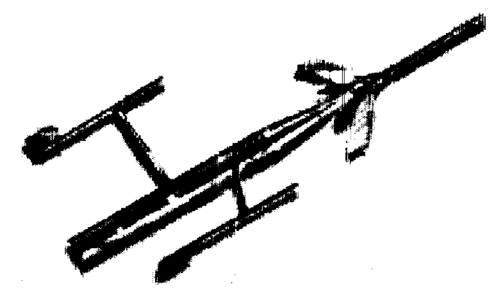


INS on a 3-axis rate table; the rate table's gimbals mirror the gimbals internal to the INS. The internal gimbals are physical devices in a traditional system, but in strapdown systems, the inertial platform is maintained computationally.

GPS/INS

- GPS & INS nicely complement one another
  - INS "dead reckons"
  - GPS does "fixes"
- Integration trade
  - *Tightly coupled*: GPS aids INS by solving for drift and misalignment, INS aids GPS tracking loop and acquisition process
  - *Loosely coupled*: little or no aiding; independent sensors in one box

Air Data Systems



- Barometric altitude
- Dynamic pressure
- Angle of attack
- Not commonly used to reconstruct position and velocity, but useful for estimating aerodynamic parameters
- Historical note: before GPS, and even until SA was turned off, use of baro altimetry was more common (e.g. Space Shuttle)

RADAR & LIDAR

- RADAR & LIDAR systems primarily provide two types of data useful for trajectory reconstruction
  - Altitude (relative to terrain, so terrain database is required)
  - Groundspeed, via Doppler shift of signals reflected off the surface, typically in combination with an INS

TACAN

- Tactical Air Navigation signals are currently broadcast by a large network of fixed stations
  - Range
  - Magnetic bearing
  - Range rate
- Only range is typically accurate enough to use in trajectory reconstruction
- The DoD plans to eventually phase out the TACAN network

Trajectory Reconstruction Examples

- Ch. 10 in R. M. Rogers' *Applied Mathematics in Integrated Navigation Systems,* has several detailed examples
  - Optical tracking from ground sites
  - Radar tracking from ground sites
  - TACAN/INS

Summary

- The filter/smoother combination provides the most accurate means of trajectory reconstruction
- Not all parameters of interest can be determined from a given flight test data set: *need to check observability*
- A variety of ground & onboard sensors may be used; trend appears to be toward increasing reliance on onboard GPS

Bibliography

- Brown, R.G. and P.Y.C. Hwang. *Introduction to Random Signals and Applied Kalman Filtering*, Wiley, NY, 1983.
- Chen, C-T. *Linear System Theory and Design*, Holt, Rinehart, and Winston, New York, 1984.
- Gelb, A. (ed.). *Applied Optimal Estimation*, The M.I.T. Press, Cambridge, MA, 1974.
- Jazwinski, A.H. *Stochastic Processes and Filtering Theory*. Academic Press, NY, 1970.
- Maybeck, P.S. *Stochastic Models, Estimation, and Control, Vol. 1*. Academic Press, NY, 1979 (out of print; reprint is available from NavTech, Alexandria, VA).
- Rogers, R.M. *Applied Mathematics in Integrated Navigation Systems*, AIAA Education Series, AIAA, Washington, DC, 2000.